Heine[4] and Suzuki[5] consider the equation which one would obtain from (6a) by inserting (8), setting $\nabla T=\nabla n=\delta f(\mathbf{\varrho}',t')=0$, and removing the factor $\partial f^0/\partial \mathcal{E}$ from under the integral sign. Clearly, the last operation leaves one with a solution of the *linearized* time-dependent Boltzmann equation, but not of the exact equation. Heine claims to prove that it is a solution of the exact equation, but his proof contains an error; however, if $\partial f^0/\partial \mathcal{E}$ is put back under the integral sign, then his proof goes through verbatim. Suzuki uses this approximate equation to discuss boundary value problems, and provides references to other recent work based on this equation. Budd[2] shows (for a special case) that the equation obtained from (7a) by inserting (8), setting $\nabla T=\nabla n=0$, and removing the factor $\partial f^0/\partial \mathcal{E}$ from under the integral sign is a solution of the linearized time-*independent* Boltzmann equation, but not of the exact equation.

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Discussion of the Dynamical Equations for the Regge Parameters*

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The dynamical equations for the Regge parameters in the relativistic, many-channel case are reduced into a single integral equation for $\mathrm{Im}\,\alpha(t)$, which is convenient to solve numerically. The solution of this integral equation is shown to exist and to be unique, after one subtraction constant for each of the functions $\alpha(t)$ and $r_{ij}(t)$, and the location of zeros of the residue functions $r_{ij}(t)$ are supplied, provided that some conditions on the subtraction constants are satisfied.

I. INTRODUCTION

It has been shown[1] that the analytic property of the Regge parameters, together with the unitarity condition, constitutes a set of equations for determining these parameters. However, many features of this set of equations, in particular, the question of what we should put in and what we can get out of them, were not well understood at that time. Neither was it realized then that inelastic two-particle intermediate states in the unitarity condition can be included, without the complication of solving some coupled integral equations.

In this paper, we shall show: (1) that the equations, with all two-particle intermediate states in the unitarity condition taken into account, can be reduced to a single integral equation which has $\mathrm{Im}\alpha(t)$ as the only unknown variable,[2] and can be solved numerically—results will be reported in a forthcoming paper[3]; (2) that the parameters of this equation will be completely specified if one subtraction constant for $\alpha(t)$ and for each of the residue functions $r_{ij}(t)$, as well as the location of zeros for $r_{ij}(t)$, are supplied; (3) that this integral equation has a unique solution if some conditions on the subtraction constants are satisfied.

These conclusions show, firstly, that the number of subtractions is not arbitrary. If we put too many restrictions on the Regge parameters by making too many subtractions, no solution for $\mathrm{Im}\alpha(t)$ would exist, while if we make too few subtractions, the solution for $\mathrm{Im}\alpha(t)$ would not be unique. Secondly, the location of the zeros of $r_{ij}(t)$ cannot be determined dynamically, but have to be supplied as input parameters. Therefore, the fact that $r_{ij}(t)$ of the Pomeranchuk trajectory vanishes at the point $\alpha_p=0$ does not follow as a dynamical consequence of our equation, but is a boundary condition itself. Whether the zeros of $r_{ij}(t)$ can be determined, once the approximate unitarity condition used here is replaced by the exact form, still awaits investigation. However, it is a consequence of analyticity and factorization for $r_{ij}(t)$ that all $r_{ij}(t)$ of the same trajectory should have the same zeros, if the possibility of double zero is ignored. The factorization law gives[4]

$$r_{ij}(t)r_{ji}(t)=r_{ii}(t)r_{jj}(t)\,,$$

and if time-reversal invariance holds,

$$r_{ij}(t)=r_{ji}(t)\,,$$

then

$$r_{ij}(t)=[r_{ii}(t)r_{jj}(t)]^{1/2}. \tag{1}$$

If $r_{ii}(t)$ has a first-order zero at z_0 and $r_{jj}(t)$ does not, then z_0 is a square-root branch point for $r_{ij}(t)$, in contradiction of the analytic property of $r_{ij}(t)$. Therefore, we should put in the same zeros for all $r_{ij}(t)$ in the dynamical equations.

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II. DERIVATION OF THE INTEGRAL EQUATION FOR Imα

The unitarity condition for $B_{ij}(l,t)$ takes the form

$$\frac{B_{ij}(l,t)-B_{ij}^*(l^*,t)}{2i}=\sum_k \frac{q_k^{2l+1}}{w}B_{ki}^*(l^*,t)B_{kj}(l,t)\theta(t-T_k)\,,\quad t>T_1,\quad (2)$$

where $B_{ij}(l,t)$ is related to the partial-wave amplitude $A_{ij}(l,t)$ by

$$B_{ij}(l,t)=(q_iq_j)^{-l}A_{ij}(l,t)\,.$$

In the above, T_k denotes the threshold of state k with T_1 the lowest threshold, q_k is the c.m. momentum of each of the particles in state k and $w=\frac{1}{2}\sqrt{t}$. All states k with the same set of conserved quantum numbers as i and j are included in the sum of (2).

Since

$$\begin{aligned}(q_k^{l*})^*&=q_k^l \quad &\text{if}\quad q_k=|q_k|\,,\\ &=e^{-i\pi l}q_k^l \quad &\text{if}\quad q_k=i|q_k|\,,\end{aligned}$$

and since q_k is real or imaginary according as $t\gtrless T_k$, we have

$$B_{ij}^*(l^*,t)=e^{i\pi l[2-\theta(t-T_i)-\theta(t-T_j)]}(q_iq_j)^{-l}A_{ij}^*(l^*,t)\,,\quad (3)$$

where $\theta(x)=1$ if $x>0$, $\theta(x)=0$ if $x<0$. Thus, (2) takes the form

$$\frac{A_{ij}(l,t)-e^{i\pi l[2-\theta(t-T_i)-\theta(t-T_j)]}A_{ij}^*(l^*,t)}{2i}=e^{i\pi l[1-\theta(t-T_i)]}\sum_k(q_k/w)A_{ki}^*(l^*,t)A_{kj}(l,t)\theta(t-T_k)\,,\quad t>T_1.\quad (4)$$

Taking the residue of both sides of (4) at $l=\alpha(t)$, we have

$$\frac{r_{ij}(t)}{2i}=e^{i\pi\alpha(t)[1-\theta(t-T_i)]}\sum_k(q_k/w)\times A_{ki}^*(\alpha^*(t),t)r_{kj}(t)\theta(t-T_k)\,,\quad t>T_1.\quad (5)$$

As before,[1] we make the approximation

$$A_{ij}(\alpha^*,t)=\frac{r_{ij}(t)}{\alpha^*(t)-\alpha(t)}=-\frac{r_{ij}(t)}{2i\,\mathrm{Im}\alpha(t)}\,,$$

good when Im$\alpha(t)$ is small. We obtain from (5)

$$r_{ij}(t)=\frac{e^{i\pi\alpha(t)[1-\theta(t-T_i)]}}{w\,\mathrm{Im}\alpha(t)}\sum_k q_kr_{ki}^*(t)r_{kj}(t)\theta(t-T_k)\,,\quad t>T_1.\quad (6)$$

From (6) we get

$$r_{ii}(t)=r_{ii}^*(t)\qquad \text{for}\quad t>T_i\,,\quad (7a)$$

$$r_{ii}(t)=e^{i\pi[\alpha(t)+\alpha^*(t)]}r_{ii}^*(t)\quad \text{for}\quad T_i>t>T_1.\quad (7b)$$

Now $r_{ii}(t)q_i^{-2\alpha}$ is an analytic function of t cut from T_1 to ∞, satisfying[1]

$$r_{ii}(t-i\epsilon)=r_{ii}^*(t)e^{2\pi i\alpha^*(t)}\quad\text{for}\quad t>T_1.\quad (8)$$

Therefore, (7) and (8) together give

$$r_{ii}(t-i\epsilon)=r_{ii}(t)e^{2\pi i\alpha^*(t)}\quad\text{for}\quad t>T_i\,,\quad (9a)$$

$$r_{ii}(t-i\epsilon)=r_{ii}(t)e^{2\pi\,\mathrm{Im}\alpha(t)}\quad\text{for}\quad T_i>t>T_1.\quad (9b)$$

We now introduce the functions $F_{ii}(t)$ and $U_{ii}(t)$ so that

$$q_i^{-2\alpha(t)}r_{ii}(t)=F_{ii}(t)e^{U_{ii}(t)}\,,\quad (10)$$

where we choose $F_{ii}(t)$ an entire function of t; then (9) and (10) give

$$U_{ii}(t+i\epsilon)-U_{ii}(t-i\epsilon)=-2i\,\mathrm{Im}\alpha(t)\ln(q_i^2)\,,\quad t>T_i\quad (11)$$

and

$$\begin{aligned}U_{ii}(t+i\epsilon)-U_{ii}(t-i\epsilon)&=-2i\,\mathrm{Im}\alpha(t)\ln(q_i^2)-2\pi\,\mathrm{Im}\alpha(t)\,,\quad T_i>t>T_1\quad (12)\\ &\approx-2i\,\mathrm{Im}\alpha(t)\ln(q_i^2)\end{aligned}$$

in the approximation that Im$\alpha(t)$ is small. From (11) and (12) we get

$$U_{ii}(t)=-\frac{(t-t_0)}{\pi}\int_{T_1}^{\infty}\frac{\ln(q_i'^2)\,\mathrm{Im}\alpha(t')}{(t'-t)(t'-t_0)}dt'\,,\quad (13)$$

where t_0 is an arbitrary point where we may wish to make a subtraction. Since $\alpha(t)$ satisfies the dispersion relation[1]

$$\alpha(t)=\alpha_0+\frac{t-t_0}{\pi}\int_{T_1}^{\infty}\frac{\mathrm{Im}\alpha(t')}{(t'-t)(t'-t_0)}dt'\,,\quad (14)$$

substituting (13) and (14) into (10) gives

$$r_{ii}(t)=F_{ii}(t)q_i^{2\alpha_0}\times\exp\left[-\frac{(t-t_0)}{\pi}\int_{T_1}^{\infty}\frac{\ln(q_i'^2/q_i^2)\,\mathrm{Im}\alpha(t')}{(t'-t)(t'-t_0)}dt'\right].\quad (15)$$

We shall now determine $F_{ii}(t)$. We require that $r_{ii}(t)$ vanishes as $|t|\to\infty$, then since

$$\begin{aligned}r_{ii}(t)&\to F_{ii}(t)q_i^{2\alpha_0}\exp\left[-\frac{\ln(q_i^2)}{\pi}\int_{T_1}^{\infty}\frac{\mathrm{Im}\alpha(t')}{t'-t_0}dt'\right]\\ &\to F_{ii}(t)t^{\alpha(\infty)}\,,\quad |t|\to\infty\,.\end{aligned}$$

$F_{ii}(t)$ is a polynomial of order n satisfying

$$n+\alpha(\infty)<0\,,\quad (16)$$

and can be written as

$$F_{ii}(t)=F_{ii}(t_0)\prod_{m=1}^{n}\left(\frac{t-z_m}{t_0-z_m}\right).\quad (17)$$

Since (15) gives

$$r_{ii}(t_0)=F_{ii}(t_0)q_{i0}^{2\alpha_0}\,,$$

we can write (15) in the form

$$r_{ii}(t)=r_{ii}(t_0)\left(\frac{q_i}{q_{i0}}\right)^{2\alpha_0}\cdot\prod_{m=1}^{n}\left(\frac{t-z_m}{t_0-z_m}\right)$$
$$\times\exp\left[-\frac{t-t_0}{\pi}\int_{T_1}^{\infty}\frac{\ln(q_i'^2/q_i^2)}{t'-t}\frac{\operatorname{Im}\alpha(t')}{t'-t_0}dt'\right],\quad(18)$$

where

$$q_{i0}=q_i(t_0).$$

The points z_m are the zeros of $r_{ii}(t)$. Take $i=j=1$ in (6), then we have

$$\operatorname{Im}\alpha(t)=\frac{1}{w}\sum_k q_k\frac{r_{k1}^*(t)r_{k1}(t)}{r_{11}(t)}\theta(t-T_k),\quad t>T_1.\quad(19)$$

Since $r_{kk}(t)$ and $r_{11}(t)$ are both real for $t>T_1$, from the factorization law (1), $r_{k1}(t)$ is real, and (19) becomes

$$\operatorname{Im}\alpha(t)=\frac{1}{w}\sum_k q_k\frac{r_{k1}^2(t)}{r_{11}(t)}\theta(t-T_k)$$
$$=\frac{1}{w}\sum_k q_k r_{kk}(t)\theta(t-T_k),\quad t>T_1.\quad(20)$$

Substituting (18) into (20) and remembering that all $r_{kk}(t)$ have the same zeros, we finally obtain the integral equation

$$\operatorname{Im}\alpha(t)=\frac{1}{w}\prod_{m=1}^{n}\left(\frac{t-z_m}{t_0-z_m}\right)\sum_k r_{kk}(t_0)q_k\left(\frac{q_k}{q_{k0}}\right)^{2\alpha_0}$$
$$\times\exp\left[-\frac{t-t_0}{\pi}\int_{T_1}^{\infty}dt'\frac{\ln(q_k'^2/q_k^2)}{t'-t}\frac{\operatorname{Im}\alpha(t')}{t'-t_0}\right]\theta(t-T_k),$$
$$t>T_1.\quad(21)$$

Once the parameters z_m, $r_{kk}(t_0)$, and α_0 are given, we can solve (21) for $\operatorname{Im}\alpha(t)$, whose existence and uniqueness will be discussed in the next section. After $\operatorname{Im}\alpha(t)$ is obtained, we can obtain $r_{ii}(t)$ from (18), and $r_{ij}(t)$, $i\neq j$, from factorization. Equation (21) is being solved numerically and the results will be reported in a forthcoming paper.[3]

III. EXISTENCE AND UNIQUENESS OF THE SOLUTION

In this section we shall investigate the existence and uniqueness of the solution of (21). For simplicity of argument, we shall discuss (21) in the absence of inelastic channels. We shall also assume that both of the particles have unit mass and that $r(t)$ has no zeros. As long as we include only finite numbers of inelastic channels in (21), the integral equation can be discussed in the same way as when inelastic channels are absent. In the potential scattering case, $r(t)$ of the leading trajectory always does not have any zeros. That the assumption of unit mass only simplifies the writing and does not change any of the conclusions is obvious.

After these simplifications have been made, Eq. (21) takes the form

$$\operatorname{Im}\alpha(\nu)=r(\nu_0)\left(\frac{\nu}{\nu_0}\right)^{\alpha_0}\left(\frac{\nu}{\nu+1}\right)^{1/2}$$
$$\times\exp\left[-\frac{(\nu-\nu_0)}{\pi}\int_0^{\infty}d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\frac{\operatorname{Im}\alpha(\nu')}{\nu'-\nu_0}\right],\quad(22)$$

where

$$\nu=\tfrac{1}{4}t-1.$$

Consider the limit $\nu\to 0$; then

$$\operatorname{Im}\alpha(\nu)\to\frac{r(\nu_0)}{\nu_0^{\alpha_0}}\nu^{\alpha_0+\frac{1}{2}}\exp\left[-\frac{\nu_0}{\pi}\int_0^{\infty}d\nu'\frac{\operatorname{Im}\alpha(\nu')}{\nu'(\nu'-\nu_0)}\ln\nu\right]$$
$$=\frac{r(\nu_0)}{\nu_0^{\alpha_0}}\nu^{\alpha(0)+\frac{1}{2}},\quad(23)$$

where

$$\alpha(0)=\alpha_0-\frac{\nu_0}{\pi}\int_0^{\infty}\frac{\operatorname{Im}\alpha(\nu')}{\nu'(\nu'-\nu_0)}d\nu'\quad(24)$$

is the value of $\alpha(\nu)$ at $\nu=0$. A necessary condition for the existence of a solution for (22) is then

$$\alpha(0)>-\tfrac{1}{2}.\quad(25)$$

For if (25) is not satisfied, then $\operatorname{Im}\alpha(\nu)$ would become infinite at $\nu=0$, which, in turn, implies, according to (24), that $\alpha(0)=\infty$, and (23) then shows that $\operatorname{Im}\alpha(\nu)$ diverges too fast for the integral in (22) to converge. Similarly, taking the limit $\nu\to\infty$, we obtain

$$\operatorname{Im}\alpha(\nu)\to\frac{r(\nu_0)}{\nu_0^{\alpha_0}}\nu^{\alpha_0}\exp\left[-\frac{1}{\pi}\int_0^{\infty}d\nu'\frac{\operatorname{Im}\alpha(\nu')}{\nu'-\nu_0}\ln\nu\right]$$
$$=\frac{r(\nu_0)}{\nu_0^{\alpha_0}}\nu^{\alpha(\infty)},\quad(26)$$

where

$$\alpha(\infty)=\alpha_0-\frac{1}{\pi}\int_0^{\infty}\frac{\operatorname{Im}\alpha(\nu')}{(\nu'-\nu_0)}d\nu'\quad(27)$$

is the value of $\alpha(\nu)$ at $\nu=\infty$. Another necessary condition for the existence of a solution for (22) is then

$$\alpha(\infty)<0.\quad(28)$$

Let us now define

$$W(\nu)=-\frac{(\nu-\nu_0)}{\pi}\int_0^{\infty}d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\frac{\operatorname{Im}\alpha(\nu')}{\nu'-\nu_0}d\nu',\quad(29)$$

then

$$\operatorname{Im}\alpha(\nu)=r(\nu_0)\left(\frac{\nu}{\nu_0}\right)^{\alpha_0}\left(\frac{\nu}{\nu+1}\right)^{1/2}e^{W(\nu)}.\quad(30)$$

Substituting (30) into (29), we obtain

$$W(\nu)=-\frac{(\nu-\nu_0)}{\pi}\frac{r(\nu_0)}{\nu_0{}^{\alpha_0}}\int_0^\infty d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\times\frac{(\nu')^{\alpha_0+\frac{1}{2}}}{(\nu'+1)^{1/2}}\frac{e^{W(\nu')}}{\nu'-\nu_0}d\nu',\quad(31)$$

which is a nonlinear integral equation for $W(\nu)$.

Let us first take the subtraction point ν_0 at infinity. Then (31) and (30) take the form

$$W(\nu)=-\frac{g^2}{\pi}\int_0^\infty d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\frac{\nu'^{\alpha(\infty)+\frac{1}{2}}}{(\nu'+1)^{1/2}}e^{W(\nu')}d\nu'\quad(32)$$

and

$$\mathrm{Im}\alpha(\nu)=g^2\frac{\nu^{\alpha(\infty)+\frac{1}{2}}}{(\nu+1)^{1/2}}e^{W(\nu)},\quad(33)$$

where

$$g^2=\lim_{\nu_0\to\infty}\frac{r(\nu_0)}{\nu_0{}^{\alpha_0}}.$$

From (33) we see that to make $\mathrm{Im}\alpha(\nu)$ positive, g^2 has to be positive. Now let us investigate under what conditions (32) would have a solution, and whether it is unique.

We can immediately conclude that if the subtraction constant $\alpha(\infty)$ does not satisfy (28), (32) does not have a solution. Now let us assume $\alpha(\infty)$ is in the range

$$-\tfrac{3}{2}<\alpha(\infty)<0,\quad(34)$$

in order that we can iterate (32) by putting $W_0(\nu)=0$, and

$$W_n(\nu)=-\frac{g^2}{\pi}\int_0^\infty d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\frac{(\nu')^{\alpha(\infty)+\frac{1}{2}}}{(\nu'+1)^{1/2}}e^{W_{n-1}(\nu')}d\nu'.\quad(35)$$

Since $[\ln(\nu'/\nu)]/(\nu'-\nu)>0$ for all ν', the right side of (35) is always negative. We therefore have

$$W_1(\nu)<W_0(\nu)=0.\quad(36)$$

Since

$$W_1(\nu)=-\frac{g^2}{\pi}\int_0^\infty d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\frac{(\nu')^{\alpha(\infty)+\frac{1}{2}}}{(\nu'+1)^{1/2}}e^{W_0(\nu')}d\nu',$$

$$W_2(\nu)=-\frac{g^2}{\pi}\int_0^\infty d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\frac{(\nu')^{\alpha(\infty)+\frac{1}{2}}}{(\nu'+1)^{1/2}}e^{W_1(\nu')}d\nu',$$

then comparing the right sides of the above two equations and making use of (36) tells us that

$$W_2(\nu)>W_1(\nu),$$

which again implies

$$W_3(\nu)<W_2(\nu).$$

This process can be continued indefinitely and we obtain

$$W_1(\nu)<W_3(\nu)<\cdots<W_{2n+1}(\nu)<\cdots<W_{2n}(\nu)\cdots<W_2(\nu)<W_0(\nu),\quad(37)$$

and the iteration sequence oscillates with smaller and smaller amplitudes. Furthermore, as can be seen from the right side of (32), any solution $W(\nu)$ of (32) is negative. Thus,

$$W(\nu)<W_0(\nu).\quad(38)$$

This implies, in turn,

$$W(\nu)>W_1(\nu),$$

and we have, in general,

$$W_{2n+1}(\nu)>W(\nu)>W_{2m}(\nu)\quad(39)$$

for all n and m. Now the sequence $W_{2n+1}(\nu)$ is a monotonically increasing function bounded above by $W_0(\nu)$ and hence tends to a limit $W^{(1)}(\nu)$ as $n\to\infty$. Similarly, the sequence $W_{2n}(\nu)$ is a monotonically decreasing function bounded below by $W_1(\nu)$ and hence tends to a limit $W^{(2)}(\nu)$ as $n\to\infty$. From (39) we have, for all solutions $W(\nu)$ of (32), the inequality

$$W^{(1)}(\nu)\geq W(\nu)\geq W^{(2)}(\nu).\quad(40)$$

All solutions of (32) are therefore bounded in between by $W^{(1)}(\nu)$ and $W^{(2)}(\nu)$. If we can show that

$$W^{(1)}(\nu)=W^{(2)}(\nu),$$

then $W^{(1)}(\nu)$ is the *unique* solution for (32).

Similarly, if we take $\nu_0=0$, then (31) and (30) become

$$W(\nu)=-\frac{r}{\pi}\nu\int_0^\infty d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\frac{(\nu')^{\alpha(0)-\frac{1}{2}}}{(\nu'+1)^{1/2}}e^{W(\nu')}\quad(41)$$

and

$$\mathrm{Im}\alpha(\nu)=r\frac{\nu^{\alpha(0)+\frac{1}{2}}}{(\nu+1)^{1/2}}e^{W(\nu)},$$

with

$$r=\lim_{\nu_0\to0}\frac{r(\nu_0)}{\nu_0{}^{\alpha_0}}.$$

Again, we require r to be positive so that $\mathrm{Im}\alpha(\nu)$ is positive. If the subtraction constant $\alpha(0)$ is less than $-\frac{1}{2}$, no solution for (41) exists, and if α_0 is in the range

$$-\tfrac{1}{2}<\alpha(0)<1,\quad(42)$$

then we can iterate and the iteration sequence behaves exactly as before, and the solutions of (41) are all bounded below and above by corresponding functions.

To prove that $W^{(1)}(\nu)$ and $W^{(2)}(\nu)$ for (32) are equal, so that the integral equation possesses a unique solution, we shall need stronger conditions on $\alpha(\infty)$ and g^2. We have from (35)

$$|W_n(\nu)-W_{n-1}(\nu)|\leq\frac{g^2}{\pi}\int_0^\infty d\nu'\frac{\ln(\nu'/\nu)}{\nu'-\nu}\frac{(\nu')^{\alpha(\infty)+\frac{1}{2}}}{(\nu'+1)^{1/2}}\times|W_{n-1}(\nu')-W_{n-2}(\nu')|d\nu',\quad(43)$$

since

$$|e^x-e^y|\leq|x-y|$$

if x and y are both negative. Applying the Schwartz inequality, we obtain from (43)

$$|W_n(\nu)-W_{n-1}(\nu)| \leq \frac{g^2}{\pi}\left\{\int_0^\infty d\nu'\left[\frac{\ln(\nu'/\nu)}{\nu'-\nu}\right]^2(\nu')^{2\alpha(\infty)+1}\right\}^{1/2} \times\left\{\int_0^\infty d\nu'\frac{[W_{n-1}(\nu')-W_{n-2}(\nu')]^2}{\nu'+1}\right\}^{1/2}. \quad (44)$$

Define

$$A(\nu)=\frac{g^2}{\pi}\left\{\int_0^\infty d\nu'\frac{[\ln(\nu'/\nu)]^2}{(\nu'-\nu)^2}(\nu')^{2\alpha(\infty)+1}\right\}^{1/2},$$

$$a=\left[\int_0^\infty \frac{A^2(\nu)}{\nu+1}d\nu\right]^{1/2};$$

then from (44) we get

$$|W_n(\nu)-W_{n-1}(\nu)| \leq A(\nu)\left\{\int_0^\infty d\nu'\frac{[W_{n-1}(\nu')-W_{n-2}(\nu')]^2}{\nu'+1}\right\}^{1/2} \leq aA(\nu)\left\{\int_0^\infty d\nu'\frac{[W_{n-2}(\nu')-W_{n-3}(\nu')]^2}{\nu'+1}\right\}^{1/2}\leq\cdots \leq a^{m-1}A(\nu)\left\{\int_0^\infty d\nu'\frac{[W_{n-m}(\nu')-W_{n-m-1}(\nu')]^2}{\nu'+1}\right\}^{1/2}. \quad (45)$$

Hence,

$$\lim_{n\to\infty}|W_n(\nu)-W_{n-1}(\nu)|=0,$$

if

$$a<1$$

or, equivalently, if

$$\left(\frac{g^2}{\pi}\right)^2\int_0^\infty d\nu\int_0^\infty d\nu'\frac{[\ln(\nu'/\nu)]^2(\nu')^{2\alpha(\infty)+1}}{(\nu'-\nu)^2(\nu+1)}<1. \quad (46)$$

The integral in (46) converges if

$$-\tfrac{1}{2}<\alpha(\infty)<0; \quad (47)$$

hence, we conclude that if $\alpha(\infty)$ satisfies (47) and g^2 is small enough to satisfy (46), the integral equation (32) possesses a unique solution.

Similarly, the integral equation (41) possesses a unique solution, if the conditions

$$-\tfrac{1}{2}<\alpha(0)<0, \quad (48)$$

$$\frac{r_0^2}{\pi^2}\int_0^\infty d\nu\int_0^\infty d\nu'\frac{[\ln(\nu'/\nu)]^2(\nu')^{2\alpha(0)}\nu}{(\nu+1)(\nu'-\nu)^2}<1, \quad (49)$$

are both satisfied. The proof will be omitted.

It appears that conditions (46), (47) or (48), (49) are stronger than necessary to insure the existence of a unique solution. Numerical solutions for the integral equation have always been found for a wide range of subtraction constants.[3] On the other hand, it probably is not true that (28) alone is sufficient to guarantee that a unique solution for (32) exists. For if $\alpha(\infty)$ is too negative, say, -10^{10}, and if g^2 is of the order of unity, it is difficult to conceive that we can obtain a solution satisfying (25). For the same reason, (25) alone probably does not insure that a unique solution for (41) exists. The same consideration applies when the subtraction point is at a finite energy ν_0. If α_0 is too large, it probably is difficult to obtain a solution satisfying $\alpha(\infty)<0$, and if α_0 is too negative, it is difficult to obtain a solution satisfying $\alpha(0)>-\frac{1}{2}$. For a physical Regge trajectory, whose subtraction parameters are reasonable, a unique solution can perhaps be expected.

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